ACTA DIDACTICA NAPOCENSIA

Volume 11, Number 1, 2018

ON TEACHING OF GENERALIZED CATALAN NUMBERS WITH THE MAPLE´S HELP

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Abstract: In Brazil we have identified a predilection of the authors of Mathematical History books for the discussion of the fundamentals of Differential and Integral Calculus. On the other hand, when we consider the teaching of Mathematics in the school context, it is essential to know the teaching of the historical and dynamic evolution of the concepts, moreover, mathematical objects more close to the Brazilian school reality. Thus, the present work discusses the notion of Catalan numbers, including its generalization process and epistemological aspects, with the adoption of some assumptions of Didactic Engineering. The work also presents a series of matrices and combinatorial properties that can be explored with the CAS Maple wih the scope to attract a larger number of students to a teaching that shows the historical, mathematical and evolutionary epistemological aspects of the generalized Catalan numbers. In this way, it is observed that the teacher must present an extensive knowledge about the notion of "numbers" and not only of the fundamentals of Differential and Integral Calculus.

Key words: Numbers of Catalan, Professor of Mathematics, Teaching, History of Mathematics.

1. Introduction

In Brazil we record historical and epistemological knowledge as especially important for mathematics teachers. However, in a hegemonic way, the books of History of Mathematics (Boyer, 1968; Eves, 1976; Roque, 2012) usually devote greater space to the discussion of the foundations of the Infinitesimal Calculus, originating from the thought of Leibinitz and Newton in the 17^{th} century. On the other hand, certain historical knowledge remains disregarded and not divulged among the professors of Mathematics in the schools relative to the current advance of Mathematics (Alves, 2016).

Moreover, in our researches (Alves, 2016; 2017) directed to the field of initial mathematics teacher education developed in Brazil, we have developed an approach design that adopts the presuppositions of Didactics of Mathematics (Brousseau, 1986), according to the tradition of the french research, aiming the initial formation of teachers with attention to the context of the classroom. In particular, in teaching in the context of the History of Mathematics, the research methodology named Didactic Engineering can contribute in the sense of describing a way to a new approach of certain historical contents that, in general, are not available in the History books of ordinary mathematics.

Besides that, a field of research that acquires greater visibility provides the teaching of Mathematics, within the context of the historical context, from the aid of technology. In our work, we have exposed the use of GeoGebra and CAS Maple softwares with the aim of exploring certain mathematical properties neglected by the History of Mathematics books (Alves, 2017). Particularly, in the present work, we will explore certain matrices involving representations of Catalan numbers and some resulting properties en virtue some generalization and it´s evolutionary process (Bobrowski; 2015; Koshy & Salmassi; 2006; Ribenboin, 1996; Varadarajan, 2006; Shapiro, 1976; Stanley, 2015).

Nevertheless, in order to describe certain structured didactic situations, in view of the teaching and training of Mathematics teachers in Brazil, we used the perspective originated from Didactic Engineering (Margolinas & Drijvers, 2015), with emphasis on the character of teacher training and that aims at proposing a didactic transposition aiming at the contents of the history of Mathematics. Thus, in

Received 27 February 2018, accepted 12 March 2018.

Cite as: Alves, F. R. V. (2018). On teaching of Generalized Catalan Numbers with the Maple´s help, *Acta Didactica Napocensia*, 11(1), 25-40, Doi: 10.24193/Adn.11.1.3.

the following section we will describe some elements of a didactic engineering and, aiming at the formation of Mathematics teachers for the teaching in a historical context.

2. Some elements of didactic engineering

Didactic engineering emerged in France in the context of major educational reforms in the sixties and seventies. From a strong concern with the changes and paradigms of mathematics teaching, didactic engineering takes a systematic perspective of preparation for research aiming to understand the phenomena of teaching and learning at all levels (Margolinas & Drijvers, 2015). We note two aspects of didactic engineering. The first area of study dedicated, in a deeper way, to the understanding of the role and the learning of the student exposed to the teaching of Mathematics, according to the real conditions of operation of the classroom. On the other hand, we recorded the didactic engineering of second generation or didactic engineering of training in order to a deeper research about the formation and initial preparation of the Mathematics teacher (Margolinas, 2005).

We find, therefore, the emergence of the terminology Didactic Engineering - ED that, despite the evolution of its premises, was used to designate/involve a research about the *modus operandi* or as a methodology for the analysis of didactic situations. In this context of reforms, since its inception, research in DM in France was built on the recognition of the need to develop their own theoretical and conceptual frameworks. We also recall that the term Didactic Engineering designates a set of sequences of classes designed, organized and articulated in time, in a coherent way by a teacher-engineer, in order to carry out a learning project for a specific population of students. It should be emphasized that, according to the design of research design and the precise rule in a research depends strongly on an educational culture. In addition, in this case, we make a special reference to the French didactic culture (Margolinas, 2005).

In this way, we can not disregard an extended framework for the adoption of new paradigms in France, especially the paradigms coming from universities. In fact, in a context, markedly of French tradition, originated in the sixties and of development and use of several notions of engineering (Chevallard, 1982), described by Leclercq (2002, p. 75) as: social engineering, pedagogical engineering and, at a point between the two previous ones, the engineering of formation (see Figure 1). "The notion of application and use of engineering in a field of training has been frequent" (Leclerc, 2002, p.76). In figure 1, Leclerq (2002, p.80) indicates the situation and the notion of engineering of formation that derived strong impregnation with several foundations for Didactics and the formation of adults. We also divided the pedagogical triangle into 2D, whose vertices are defined by the terms: savoir, teacher (*professeur*) and student/learner *(élêve* or *apprenant*).

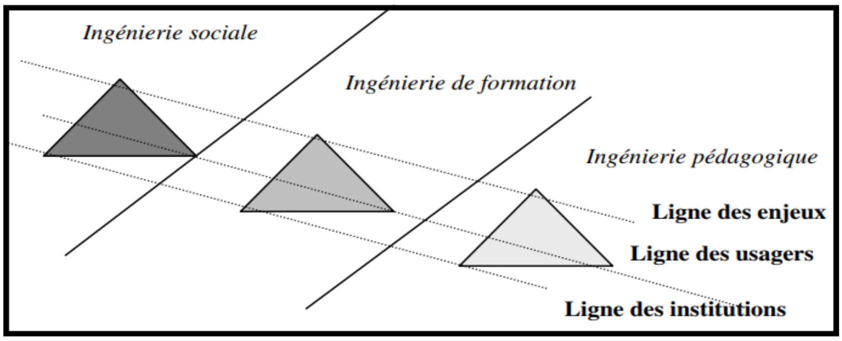

Figure 1. *Descriptive picture of the notions of engineering developed in France in the 1960* (Leclercq, 2002)

Margolinas & Drijvers (2015, p. 890) explain that "didactic engineering provides at least one existence theorem. They show that teaching is possible under certain conditions. But such conditions may be difficult to satisfy in ordinary teaching”. The authors emphasize that at that time the risk of the desired transformations in the education system was observed, and that the official teaching system itself was not prepared to integrate such transformations. Thus, in the following section, we will develop a historical and epistemological analysis aimed of the teaching Catalan numbers and a systematic way for of its exploitation by the teacher, with the help of technology. Next, we will discuss some matrix representations that allow the exploration of technology for the investigation and generalization of properties derived from mathematical induction that can be explored in the school.

3. Some historical aspects about the generalized numbers of Catalan

The Belgian mathematician Eugene C. Catalan discovered, in 1838, the numbers that acquired the greatest circulation within the mathematical studies of his time and, despite a modest initial mathematical contribution that involved the description of the following formula or formal definition $C_n = C(n) = \binom{2n}{n} - \binom{2n}{n-1}, n \geq 0$ by means of a process of determining the quantity of triangles, by different modes, circumscribed in polygonal figures. The idea comes from the problem of the classical problems of triangulation of L. Euler, presented by himself in 1751 (Koshy, 2009, p.107). Euler introduced the closed formula itself (Stanley, 2015, p.178). The following numerical list indicated by Koshy (2009) describes some of the first numbers of Catalan below:

$$\{1,\ 1,\ 2,\ 5,\ 14,\ 42\ ,132\ ,429\ ,1.430\ ,4.862\ ,16.796\ ,58.786\ ,208.012\ ,\mathrm{K}\ ,\mathrm{C}_n,\mathrm{K}\ ,\mathrm{K}\}\ (*)$$

The recent discovery by Luo Jianjin in 1988 approached and describes the first appearance of Catalan numbers due to the work of the Chinese mathematician Ming Antu (c.1692-c.1763) who wrote a book in 1731 which included some trigonometric expansions involving Catalan numbers (Stojadinovic, 2015). Stanley examines the ubiquity of Catalan numbers (2015, p.177) in the following excerpt:

> In the modern literature of Mathematics, Catalan numbers are extraordinarily ubiquitous. Although they occur under varying aspects, we have made use of mathematics with them around and it is difficult to imagine the time when they were unknown or, obscurely known and not appreciated. It may then be a surprise that Catalan numbers have a rich history and multiple discoveries, even recently. Here we have preceded a review of about 200, from its discovery to the present. (Stanley, 2015).

In 1751, Leonhard Euler (1707-1783) found a closed formula for such numbers. The mathematician Christian Goldbach (1690-1764) also confirmed some results provided by L. Euler lacking a necessary formal proof. However, only with the results of E. Catalan that the subject acquired greater popularity. In fact, the mathematician Eugene Charles Catalan (1814 - 1894) was born in Bruges, Belgium. He studied at the *École Polythenique* de Paris, occupying the simple role of repeater (Bilu, Bugeaud & Mignotte, 2010, p.1) and received, according to Koshy (2007, p.105), his doctorate in Sciences in 1841. Catalan he became a professor of mathematics at the Chalonssur-Marne College and then in France, he became professor of analysis at the Université of Liège in Belgium. He published works such as *Élements de Geometriè* and *Notions d'Astronomie*, in 1843 and 1860, respectively. In the field of advanced mathematics, he published numerous articles in the field of multiple integrals, surface theory, mathematical analysis, calculus and probabilities (Koshy, 2007; 2012).

Grimaldi (2012, p.147) recalls that Gabriel Lamé (1795-1870) was the first to provide an elegantly prove, using the models of the Combinatorial, the results introduced, without the formal treatment preliminarily required by L. Euler and L von Segner. Its results were published in some mathematical articles in the year 1838. In addition, a little later, in 1839, Catalan wrote several articles on the subject, where he determined the number of forms or paths of a chain of (n + 1) symbols with parentheses with 'n' pairs so that it can envelop such symbols (Guimarães, 2012).

Euler was in Berlin (Prussia) at that time, while his friend and former mentor Goldbach was in St Petersburg (Imperial Russia). They met for the first time when Euler arrived in St. Petersburg in 1727 as a young man, and began a lifelong friendship with 196 letters between them (Varadarajan, 2006). In September 1751, Euler wrote a letter to Goldbach communicating the unexpected discovery of a species of numbers originating from an ancient problem of triangulation of regular polygons.

In the image below we indicate some mathematicians who contributed directly or indirectly to the evolution of Catalan numbers. The Chinese mathematician Ming Antu, on the left side, then we see the image of L. Euler. Then Christian Goldbach, and finally on the right side. L. Catalan. Here we observe a clear relevance of the evolutionary process of the numbers of Catalan, in view of the progressive contribution of several mathematicians over time. This mathematical evolutionary understanding is important for the Mathematics teacher.

Figure 2. *Some mathematicians who contributed to the evolution and generalization of Catalan numbers.*

Bilu, Bugeaud & Mignotte (2010, p.2) point to one of the first theorems provided by Catalan as a repeating teacher at the *École Polytechnique*, published in 1842, without the corresponding proofs. Two years later, Catalan wrote a famous letter to the famous newspaper Crelle's Journal, indicating the need for corrections of a published article by another author (See in Figure 3).

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48. ***Théorème.*** **Deux nombres entiers consécutifs, autres que 8 et 9, ne peuvent être des puissances exactes. (Catalan.)**

13.
Note
extraite d'une lettre adressée à l'éditeur par Mr. *E. Catalan*, Répétiteur à l'école polytechnique de Paris.

„Je vous prie, Monsieur, de vouloir bien énoncer, dans votre recueil, le „théorème suivant, que je crois vrai, bien que je n'aie pas encore réussi à „le démontrer complètement: d'autres seront peut-être plus heureux:
„Deux nombres entiers consécutifs, autres que 8 et 9, ne peuvent être „des puissances exactes; autrement dit: l'équation $x^m - y^n = 1$, dans „laquelle les inconnues sont entières et positives, n'admèt qu'une seule „solution."

Figure 3. *Catalan wrote a letter indicating corrections in an important journal of Mathematics.*

We rescued a thought by Campbell (1984) that questioned the student's need to deal with a concrete and real problem in order to understand the role of Catalan numbers. To illustrate its field of application, Campbell (1984, p. 197-198) describes an imaginary dialogue between two students (see figure 4). The author seeks to mean to the reader that many problems, whose eminently theoretical origin, originated from the abstract and refined thought of mathematicians can be the object of several applications, above all, to the computational symbolic calculation. Note that Campell (1984), in the context of using Pascal, an old computational language, discusses how to obtain the factorization of the following large Catalan number:

$$C_{173} = 2^4 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31 \cdot 37 \cdot 47 \cdot 59 \cdot 61 \cdot 67 \cdot 89 \cdot 97 \cdot 101 \cdot 109 \cdot 113 \cdot 179 \cdot 181 \cdot 191 \cdot 193 \cdot 197 \cdot 199 \cdot$$
$$\cdot 211 \cdot 223 \cdot 227 \cdot 229 \cdot 233 \cdot 239 \cdot 241 \cdot 251 \cdot 257 \cdot 263 \cdot 269 \cdot 271 \cdot 273 \cdot 279 \cdot 283 \cdot 293 \cdot 307 \cdot 311 \cdot 313 \cdot 317 \cdot 331 \cdot 337\,.$$

Before concluding the current section, we will present the following important mathematical definitions that confirm an unstoppable evolutionary process of Mathematics and, in particular, an evolutionary mathematical process and generalization of Catalan numbers (Brasil Junior, 2014). In figure 4 we observe a scenario that indicates the important relations between the nuneros of Catalan and the progress of the technology that involves the use of other methods for their systematic study (Varey, 2011). From now on we will see our first formal definition.

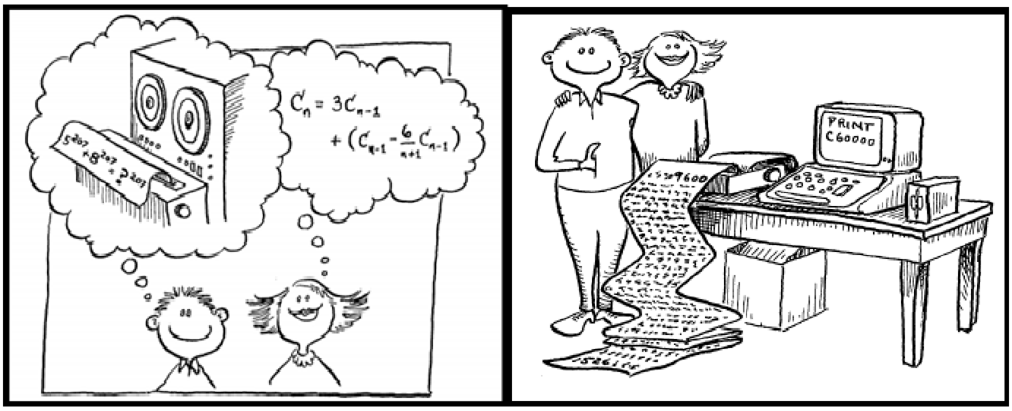

Figure 4. *Campbell (1984) dicusses the relations between Catalan numbers and computational technology.*

Definition 1: A Generalized Number of Catalan is defined by $C(n,k)=\frac{1}{kn+1}\binom{(k+1)n}{n}=$

$=\frac{1}{kn+1}\binom{kn+n}{n}=\frac{1}{kn+1}\frac{(kn+n)!}{n!(kn)!}$, where $k\geq 0, C(n,1)=C_{n,k}$. (Koshy, 2007, p.375).

Another definition of a generalized number of Catalan was provided by Gould (1972), as follows $C_n(a,b)=\frac{a}{a+bn}\binom{a+bn}{n}$. Gould (1972) verified the following unespected equality $C_n(a+c,b)=\sum_{k=0}^{n}C_k(a,b)\cdot C_{n-k}(c,b)$, which can be interpreted as a generalization of Segner's identity ($C_n=\sum_{i=0}^{n-1}C_{n-i}C_i$). Koshy (2007, p.375) mentions that we can still write a generalized number of Catalan as follows, in terms of the 'k' parameter $\frac{1}{n}\binom{(k+1)n}{n-1}$.

In fact, it is enough to employ the definition of the binomial number and write the equality $\frac{1}{n}\binom{(k+1)n}{n-1}=\frac{1}{n}\cdot\frac{(kn+n)!}{(n-1)!(kn+1)!}=\frac{1}{(kn+1)}\cdot\frac{(kn+n)!}{\text{n}!(kn)!}=\frac{1}{kn+1}\binom{kn+n}{n}$ which corresponds to the term previously defined by $C(n,k)$. Let's see that: $C(1,k)=\frac{1}{k+1}\binom{(k+1)}{1}=\frac{1}{k+1}\frac{(k+1)!}{k!}=1$,

$$C(2,k)=\frac{1}{2k+1}\binom{2(k+1)}{2}=\frac{1}{2k+1}\frac{(2k+2)!}{2(2k)!}=\frac{(2k+2)(2k+1)}{2(2k+1)}=k+1,$$

$$C(3,k)==\frac{1}{3k+1}\binom{3(k+1)}{3}=\frac{1}{3k+1}\frac{(3k+3)!}{3!(3k)!}=\frac{(3k+3)(3k+2)(3k+1)(3k)!}{3\cdot 2\cdot(3\text{k}+1)\cdot(3\text{k})!}=\frac{(k+1)(3k+2)}{2},$$

$$C(4,k)\frac{1}{4k+1}\binom{4k+4}{4}=\frac{1}{4k+1}\frac{(4k+4)!}{4!(4k)!}=\frac{1}{4k+1}\frac{(k+1)(4\text{k}+3)(4\text{k}+2)(4\text{k}+1)(4\text{k})!}{3\cdot 2\cdot(4k)!}=\frac{(k+1)(4\text{k}+3)(2\text{k}+1)}{3}.$$

Note that it becomes rather complicated to decide whether such numbers are in fact integers. In fact, we see that: $\binom{kn+n}{n}-\binom{kn+n}{n-1}=\frac{(kn+n)!}{n!(kn)!}-\frac{(kn+n)!}{(n-1)!(kn+1)!}=$

$$=\frac{(kn+n)!}{1}\left(\frac{kn+1}{n!(kn+1)!}-\frac{1}{(n-1)!(kn+1)!}\right)=\frac{(kn+n)!}{(kn+1)!}\left(\frac{kn+1}{n!}-\frac{n}{n!}\right)=\frac{(kn+n)!}{n!(kn+1)!}\left(\frac{kn+1}{1}-\frac{n}{1}\right)=$$

$=\frac{(kn+n)!}{n!(kn+1)!}\left(\frac{kn+1}{1}-\frac{n}{1}\right)=\frac{(kn+n)!}{n!(kn+1)!}\times[(k-1)n+1]$. Therefore, we have obtained $\binom{kn+n}{n}-\binom{kn+n}{n-1}=\frac{(kn+n)!}{n!(kn+1)!}[(k-1)n+1]=\frac{1}{(kn+1)}\binom{kn+n}{n}[(k-1)n+1]$. Or, we determined that $\binom{kn+n}{n}-\binom{kn+n}{n-1}=C(n,k)\cdot[(k-1)n+1]$. Now, if we notice that the whole number indicated $\binom{kn+n}{n}-\binom{kn+n}{n-1}$ implies that the element should divide the product $\binom{kn+n}{n}\times[(k-1)n+1]$. On the other hand, we can verify that the following indicated elements have no common factors, that is $MCD(kn-n+1,kn+1)=1$

In fact, if we suppose that $MCD(kn+1-n,kn+1)=d$. Thus, we can easily deduce that $d|(kn+1-n)-(kn+1)=n \leftrightarrow d\,|n$. However, we know that $d|kn+1$ and finally $d|1$. Now, taking up the previous expression involving equality $\binom{kn+n}{n}-\binom{kn+n}{n-1}=\frac{1}{(kn+1)}\binom{kn+n}{n}[(k-1)n+1]$ implies that the number of $\binom{kn+n}{n}$ is divisible by the expression $(kn+1)$ and, thus, a generalized number of Catalan $C(n,k)=\frac{1}{kn+1}\binom{(k+1)n}{n}$ is always an integer, for $\forall n,k\geq 0$.

In 1874, E. Catalan concluded that the numbers of the form $\left(\frac{(2m)!(2n)!}{m!n!(m+n)!}\right)$ are also integers and that some time later they were studied by several mathematicians (Gessel & Chin, 2006). Some time later, it appears in the specialized literature the following definition that received more attention from the work of Gessel (1992).

Definition 2: Given the integers we define $S(m,n)=\frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}=\frac{(2m)!(2n)!}{m!n!(m+n)!}$ the super numbers Catalan or the bivariate Catalan numbers. (Gessel, 1992).

In 1890 the following identity $\sum_{k=-b}^{b}(-1)^k\binom{2a}{a-k}\binom{2b}{b-k}=\frac{(2a)!(2b)!}{a!b!(a+b)!}$ was studied by the mathematician Koloman von Szily (Von Szily, 1893). A few years ago Gessel (1992) introduced the notion of super Catalan numbers. Let's see a lemma that corresponds to the important properties related to the super numbers of Catalan.

Lemma 1: For any integers $m,n\geq 0$ the following properties for the Catalan super numbers occur: (i) $S(m+1,n)+S(m,n+1)=4\cdot S(m,n)$; (ii) $S(m,n)$ is an integer, for any $m,n\geq 0$. (By the author)

Proof. In item (i), it is enough to see that $S(m+1,n)+S(m,n+1)=\frac{(2m+2)!(2n)!}{(m+1)!n!(\mathrm{m+n+1})!}+$

$$\frac{(2m)!(2n+2)!}{\mathrm{m!(n+1)!(m+n+1)!}}=\left(\frac{(2m)!(2n)!}{m!n!(m+n+1)!}\right)\times\left[\frac{(2m+2)(2m+1)}{m+1}+\frac{(2n+2)(2n+1)}{n+1}\right]=$$

$=\frac{S(m,n)}{m+n+1}[2(2m+1)+2(2n+1)]=4\cdot\frac{S(m,n)}{m+n+1}[4m+4n+4]=4\cdot S(m,n)$. For item (ii), we will use the induction model in relation to 'm'. In fact, we note $\mathrm{S}(0,n)=\frac{\binom{0}{0}\binom{2n}{n}}{\binom{n}{n}}=\binom{2n}{n}$ and the property is true. We will admit $m=m_0$ the inductive step to and, thus, we will consider equality $S(m_0+1,n)+S(m_0,n+1)=4\cdot S(m_0,n)\leftrightarrow S(m_0+1,n)=4\cdot S(m_0,n)-S(m_0,n+1)$. But from the inductive step, we know that both numbers $S(m_0,n),S(m_0,n+1)$ are integers. Follow the result. Matrix representation of certain numbers has always attracted the attention of mathematicians. Let us see a representation corresponding to the Catalan numbers. Thus, we will consider the special matrices as follows: $M_n=\begin{pmatrix}\binom{2n-1}{n-1} & \binom{2n-1}{n-1}\\ \frac{2n}{n+1} & 2\end{pmatrix}_{2x2}$, $N_n=\begin{pmatrix}\binom{2n-2}{n-1} & \binom{2n-2}{n-2}\\ \binom{2n}{n} & \binom{2n}{n-1}\end{pmatrix}$. We will show that it has an intimate relationship with Pascal's triangle. From this expression, let us see a second lemma.

Lemma 2 (Trivedi & Jha, 2017): For every $n\geq 1$ we have: (i) $\det(M_n)=C_n$; (ii) if $N_n=\begin{pmatrix}\binom{2n-2}{n-1} & \binom{2n-2}{n-2}\\ \binom{2n}{n} & \binom{2n}{n-1}\end{pmatrix}$ and $N_1=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$,

then $\det N_n=\det\begin{pmatrix}\binom{2n-2}{n-1} & \binom{2n-2}{n-2}\\ \binom{2n}{n} & \binom{2n}{n-1}\end{pmatrix}=C_n\cdot C_{n-1}$.

Proof. In fact, let us see that its determinant can be determined by $\det M_n = \det\begin{pmatrix} \binom{2n-1}{n-1} & \binom{2n-1}{n-1} \\ \frac{2n}{n+1} & 2 \end{pmatrix}_{2x2} = 2\cdot\binom{2n-1}{n-1} - \frac{2n}{n+1}\cdot\binom{2n-1}{n-1}$. Then, we will make some simplifications $2\binom{2n-1}{n-1} = 2\frac{(2n-1)!}{(n-1)!n!} = \frac{2\cdot n(2n-1)!}{n\cdot(n-1)!n!} = \frac{(2n)!}{n!\cdot n!} = \binom{2n}{n}$. On the other hand, the expression $\frac{2n}{n+1}\binom{2n-1}{n-1} = \frac{2n}{n+1}\cdot\frac{(2n-1)!}{(n-1)!n!} = \frac{(2n)!}{(n+1)n!\cdot(n-1)!} = \frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n-1}$. Finally, we will replace $\det M_n = 2\cdot\binom{2n-1}{n-1} - \frac{2n}{n+1}\cdot\binom{2n-1}{n-1} = \binom{2n}{n} - \binom{2n}{n-1} = C_n$. Now let's look at item (ii). In fact, it is sufficient to observe that $\det N_n =$

$$= \binom{2n-2}{n-1}\times\binom{2n}{n-1} - \binom{2n-2}{n-2}\times\binom{2n}{n} = \frac{(2\text{n}-2)!}{(n-1)!(n-1)!}\cdot\frac{(2n)!}{(n-1)!(n+1)!} - \frac{(2n-2)!}{(n-2)!\text{n}!}\cdot\frac{(2n)!}{n!n!} =$$

$$= (2\text{n}-2)!(2n)!\left[\frac{1}{(n-1)!(n-1)!}\cdot\frac{1}{(n-1)!(n+1)!} - \frac{1}{(n-2)!\text{n}!}\cdot\frac{1}{n!n!}\right] = (2\text{n}-2)!(2n)!\left[\frac{1}{((n-1)!)^3(n+1)!} - \frac{1}{(n-2)!(\text{n}!)^3}\right]$$

$$= \frac{(2\text{n}-2)!(2n)!}{1}\left[\frac{n^3}{((\text{n}!)^3(n+1)!} - \frac{1}{(n-2)!(\text{n}!)^3}\right] = \frac{(2\text{n}-2)!(2n)!}{(\text{n}!)^3}\left[\frac{n^3}{(n+1)!} - \frac{1}{(n-2)!}\right] =$$

$$= \frac{(2\text{n}-2)!(2n)!}{(\text{n}!)^3}\left[\frac{n^3}{(n+1)!} - \frac{n(n+1)(n-1)}{(n+1)!}\right] = \frac{(2\text{n}-2)!(2n)!}{(\text{n}!)^3(n+1)!}\left[n^3 - n(n^2-1)\right] = \frac{(2\text{n}-2)!(2n)!}{(\text{n}!)^3(n+1)!}\cdot\frac{n}{1}$$

$$= \frac{(2\text{n}-2)!(2n)!}{\text{n!n!n!}(n+1)!}\cdot\frac{n}{1} = \frac{(2\text{n}-2)!(2n)!}{(\text{n}-1)!\text{n!n!}(n+1)!} = \frac{(2n-2)!}{(n-1)!\text{n}!}\cdot\frac{(2n)!}{n!(\text{n}+1)!} = C_{n-1}\cdot C_n, \forall n \geq 0.$$

Let us now consider the matrix indicated by $M_{n,k} = \frac{1}{kn-n+1}\begin{pmatrix} \binom{kn+n-1}{n-1} & \binom{kn+n-1}{n-1} \\ \frac{kn+n}{kn+1} & \frac{kn+n}{n} \end{pmatrix}_{2x2}$.

Then, using the relation indicated above $\binom{kn+n}{n} - \binom{kn+n}{n-1} = C(n,k)\cdot\left[(k-1)n+1\right]$ we will determine its determinant: $\det M_{n,k} = \frac{1}{kn-n+1}\det\begin{pmatrix} \binom{kn+n-1}{n-1} & \binom{kn+n-1}{n-1} \\ \frac{kn+n}{kn+1} & \frac{kn+n}{n} \end{pmatrix}_{2x2}$

Note that $\frac{kn+n}{n}\binom{kn+n-1}{n-1} = \frac{kn+n}{n}\cdot\frac{(kn+n-1)!}{(n-1)!(kn)!} = \frac{kn+n}{1}\cdot\frac{(kn+n-1)!}{n!(kn)!} = \frac{(kn+n)!}{n!(kn)!} = \binom{kn+n}{n}$

In a similar way, let's see $\frac{kn+n}{kn+1}\cdot\binom{kn+n-1}{n-1} = \frac{kn+n}{kn+1}\cdot\frac{(kn+n-1)!}{(n-1)!(kn)!} = \frac{1}{1}\cdot\frac{(kn+n)!}{(n-1)!(kn+1)!}$

$$=\binom{kn+n}{n-1}.$$

In this way, let's see that: $\det M_{n,k}=\frac{1}{kn-n+1}\times\det\begin{pmatrix}\binom{kn+n-1}{n-1} & \binom{kn+n-1}{n-1}\\ \frac{kn+n}{kn+1} & \frac{kn+n}{n}\end{pmatrix}_{2x2}=$

$$=\frac{1}{kn-n+1}\left(\frac{kn+n}{n}\cdot\binom{kn+n-1}{n-1}-\frac{kn+n}{kn+1}\cdot\binom{kn+n-1}{n-1}\right)=\frac{1}{kn-n+1}\left(\binom{kn+n}{n}-\binom{kn+n}{n-1}\right)=C(n,k).$$

From the previous argument, we will state the following own lemma 3(*).

Lemma 3(*): For any integer $C(n,k)$, n, k ≥ 1, we have that the generalized matrix $M_{n,k}=M(n,k)=\frac{1}{kn-n+1}\times\begin{pmatrix}\binom{kn+n-1}{n-1} & \binom{kn+n-1}{n-1}\\ \frac{kn+n}{kn+1} & \frac{kn+n}{n}\end{pmatrix}_{2x2}$ has determinant equal to a generalized number of Catalan C(n,k) of order n.

In the next section we will see some applications of technology for the investigation of Catalan numbers. We will list some properties that can be explored in the classroom around historical research.

4. Applications of technology for the teaching of Catalan numbers with the Maple´s help.

Now, we will explore some matrix representations related to Catalan numbers. We will verify that when we deal with the generalized numbers of Catalan the operational calculation becomes quite complicated and the use of software such as Maple can provide the exploration of an investigative process under the protection of an inductive thinking aiming at the confirmation of certain important properties. In a preliminary way, we present in the table below some particular cases of the generalized numbers of Catalan. Notice that the expressions that depend on the 'k' parameter become quite complex.

In this section we will explore some properties of the dot matrix representations that we have introduced in the past sections and we have definined by: $M_n=\begin{pmatrix}\binom{2n-1}{n-1} & \binom{2n-1}{n-1}\\ \frac{2n}{n+1} & 2\end{pmatrix}_{2x2}$,

$N_n=\begin{pmatrix}\binom{2n-2}{n-1} & \binom{2n-2}{n-2}\\ \binom{2n}{n} & \binom{2n}{n-1}\end{pmatrix}$. $M_{n,k}=\frac{1}{kn-n+1}\begin{pmatrix}\binom{kn+n-1}{n-1} & \binom{kn+n-1}{n-1}\\ \frac{kn+n}{kn+1} & \frac{kn+n}{n}\end{pmatrix}_{2x2}$. We still observe that the super Catalan number $S(m,n)=\frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}=\frac{(2m)!(2n)!}{m!n!(m+n)!}$ can be expressed by the matrix defined by $A(m,n)=\begin{pmatrix}(2m)! & 0 \\ 1 & (2n)!\end{pmatrix}\begin{pmatrix}\frac{1}{m!n!} & 0 \\ 1 & \frac{1}{(m+n)!}\end{pmatrix}=B(m,n)\cdot C(m,n)$. We easily determine that $\det A(m,n)=\mathrm{S}(\mathrm{m},\mathrm{n})$ in view of $\det A(m,n)=\det B(m,n)\cdot\det C(m,n)$.

In the table below we observe a preliminary list of the generalized numbers of Catala, dependent on a parameter. Note that apparently they are not integers, however, as we have shown in the previous section, all expressions can not be rational numbers.

n	Generalized Catalan Numbers - $C(n,k)$
5	$\frac{1}{24}(k+1)(5k+2)(5k+3)(5k+4)$
6	$\frac{1}{10}(k+1)(6k+5)(3k+2)(2k+1)(3k+1)$
7	$\frac{1}{720}(k+1)(7k+5)(7k+3)(7k+6)(7k+4)(7k+2)$
8	$\frac{1}{315}(k+1)(8k+3)(4k+3)(2k+1)(8k+7)(4k+1)(8k+5)$
9	$\frac{1}{4480}(k+1)(9k+2)(9k+4)(3k+2)(9k+8)(3k+1)(9k+5)(9k+7)$
10	$\frac{1}{4536}(k+1)(5k+1)(10k+3)(5k+2)(2k+1)(5k+3)(10k+7)(5k+4)(10k+9)$
11	$\frac{1}{3628800}(k+1)(11k+6)(11k+7)(11k+2)(11k+8)(11k+3)(11k+9)(11k+4)(11k+10)(11k+5)$
12	$\frac{1}{11550}(k+1)(12k+11)(6k+5)(4k+3)(3k+2)(12k+7)(2k+1)(12k+5)(3k+1)(4k+1)(6k+1)$
13	$\frac{1}{479001600}(k+1)(13k+11)(13k+9)(13k+7)(13k+5)(13k+3)(13k+12)(13k+10)(13k+8)(13k+6)(13k+4)(13k+2)$

With the use of the software we can provide an investigative expedient for the exploration of the numerical behavior of the matrices. We observe in Figure 5 that, for a large set of numerical data, the mathematical property is invariant. We note that the result of the determinant is always an integer.

Now, let's consider some identities introduced by Bessel (1992). The first derives from the following combinatorial expression $6\frac{(2n)!}{n!(n+2)!}$. Bessel (1992) comments that although it does not seem like such an expression will always correspond to an integer. With CAS Maple, we can investigate a large set of values and we can see that it is indeed correct. On the other hand, our investigative process aims to introduce the Catalan numbers. Gessel (1992) discussed the validity of the following identity $6\frac{(2n)!}{n!(n+2)!}=4C_n-C_{n+1}$, for all $n\geq 0$. Based on the research that seeks to understand the numerical behavior of the expression. Gessel (1992) employs a computational model with the objective of confirming certain properties related to the Catalan super numbers. In the Figure 6 below we present the numerical data that confirm the equality for a large set of particular cases.

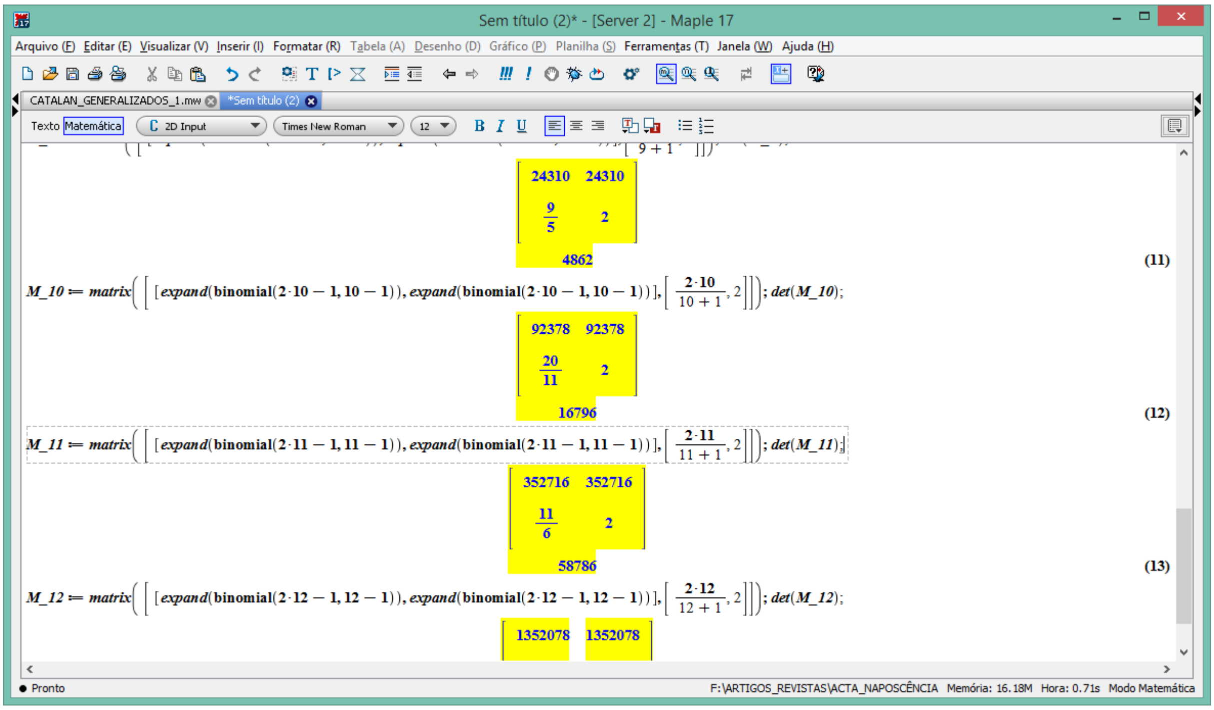

Figure 5. *With CAS Maple we can explore numerical and combinatorial relations derived from the matrix representations originated from Catalan numbers.* ***(****Source: Prepared by the author)*

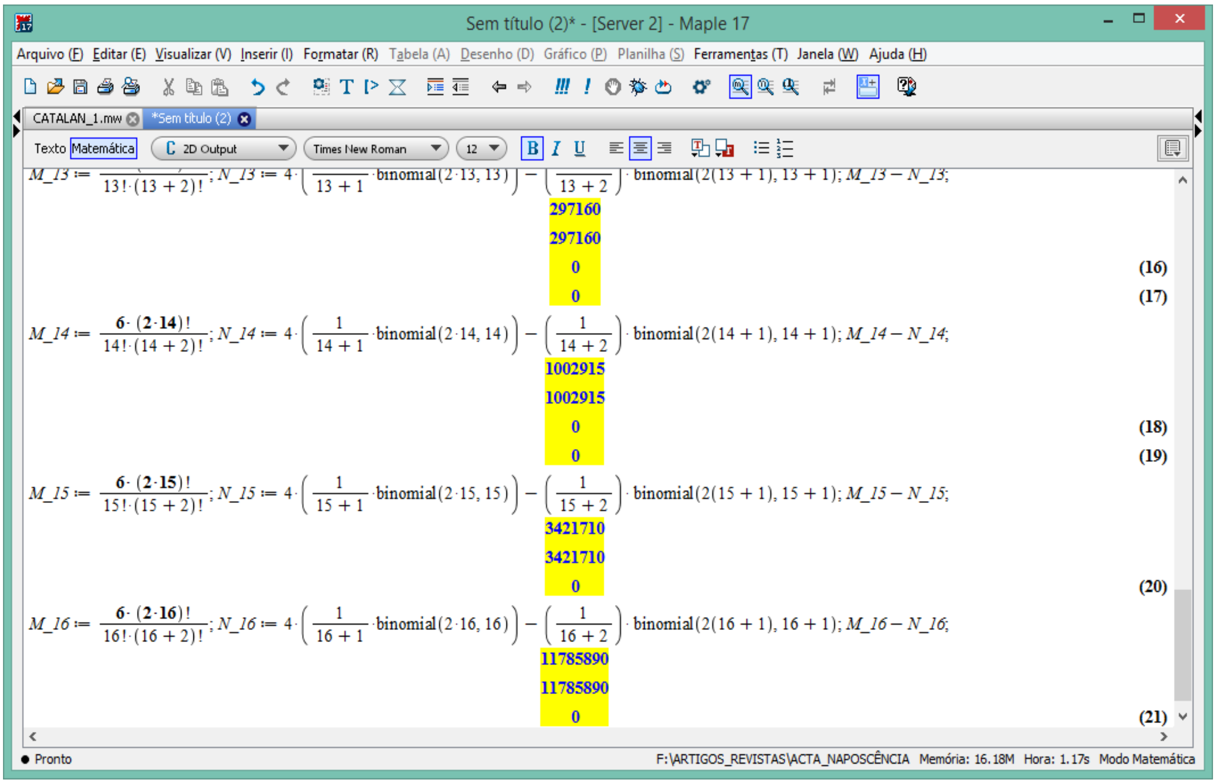

Figure 6. *With CAS Maple we can explore numerical and combinatorial relations derived from the matrix representations originated from Generalized Catalan numbers. (Source: Prepared by the author)*

We observe that the result of the determinant is always an integer number and when we evaluate the following difference $6\frac{(2n)!}{n!(n+2)!}-4C_n+C_{n+1}=0$ the software always indicates the value 0, numerical behavior indicates the validity of the identity. In our next example, we recall the high operational cost of determining Catalan numbers dependent on a 'k' parameter, especially when such values tend to grow. On the other hand, the expressions indicated in the figure below can be tested and confirm their behavior according to Lemma 2. For example, we can explore the particular numerical behavior of the following generalized numbers of Catalan, and understand from the numerical results that the resulting value is always an integer.

$$C(11,k) = \frac{1}{3628800}(k+1)(11k+6)(11k+7)(11k+2)(11k+8)(11k+3)(11k+9)\cdot$$
$$\cdot(11k+4)(11k+10)(11k+5)$$

In the context of the interpretation of the computational language, we can recall the interpretation of the numbers of Catalan through the intermediate known that the number of well-formed orderings of n open and n closed parentheses (Rubestein, 1993).

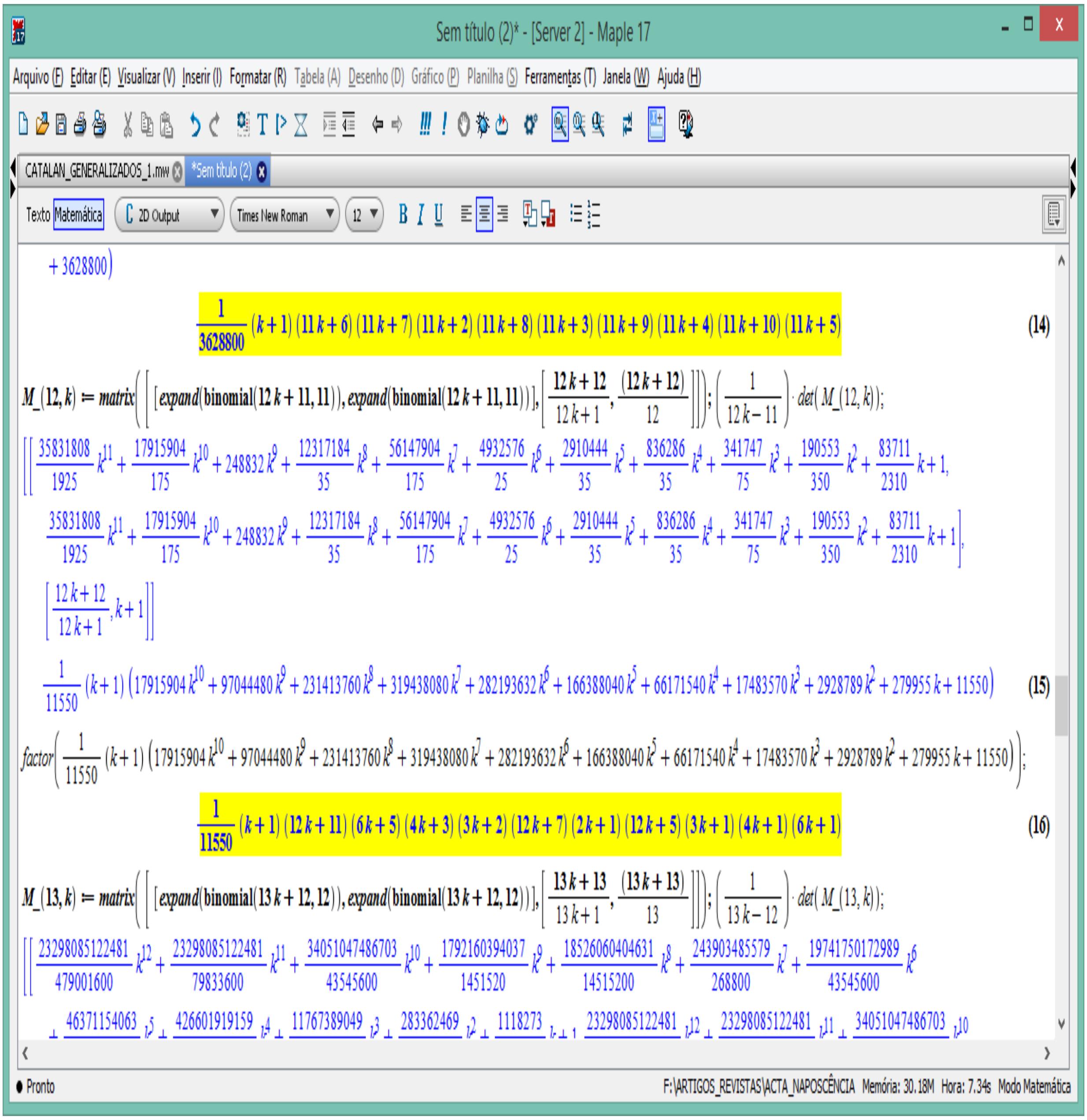

Figure 7. *With CAS Maple we can explore numerical and combinatorial relations derived from the matrix representations originated from Generalized Catalan numbers. (Source: Prepared by the author)*

Let's look at one last property that can be exploited with CAS Maple. In this sense, in the figure below, we present the behavior of the matrix determinant whose entries are, precisely, the Catalan numbers. The first matrices that we indicate are given by the list $\begin{pmatrix} C_0 & C_1 \\ C_2 & C_3 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \begin{pmatrix} C_0 & C_1 & C_2 \\ C_3 & C_4 & C_5 \\ C_6 & C_7 & C_8 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix} \cdots, \begin{pmatrix} C_m & \cdots & C_{m+n} \\ \vdots & \ddots & \vdots \\ C_{m+n} & \cdots & C_{n+n+m} \end{pmatrix}$, with $n \geq 0$ and $m \in \{0,1\}$. We observed that with increasing order of the matrix, preserving the order of the indices, the values of the determinant are always constant equal to 1. We note that, according to the authors Mays & Mays (2000, p. 131) that any finite square submatrix has a positive determinant, that is, we can verify that $\det \begin{pmatrix} C_m & \cdots & C_{m+n} \\ \vdots & \ddots & \vdots \\ C_{m+n} & \cdots & C_{n+n+m} \end{pmatrix} > 0, \forall m,n \in IN$. With the use of *software* in the investigative process we can study the behavior of the previous determinant, with increasing order, involving unexpected relations between Catalan numbers.

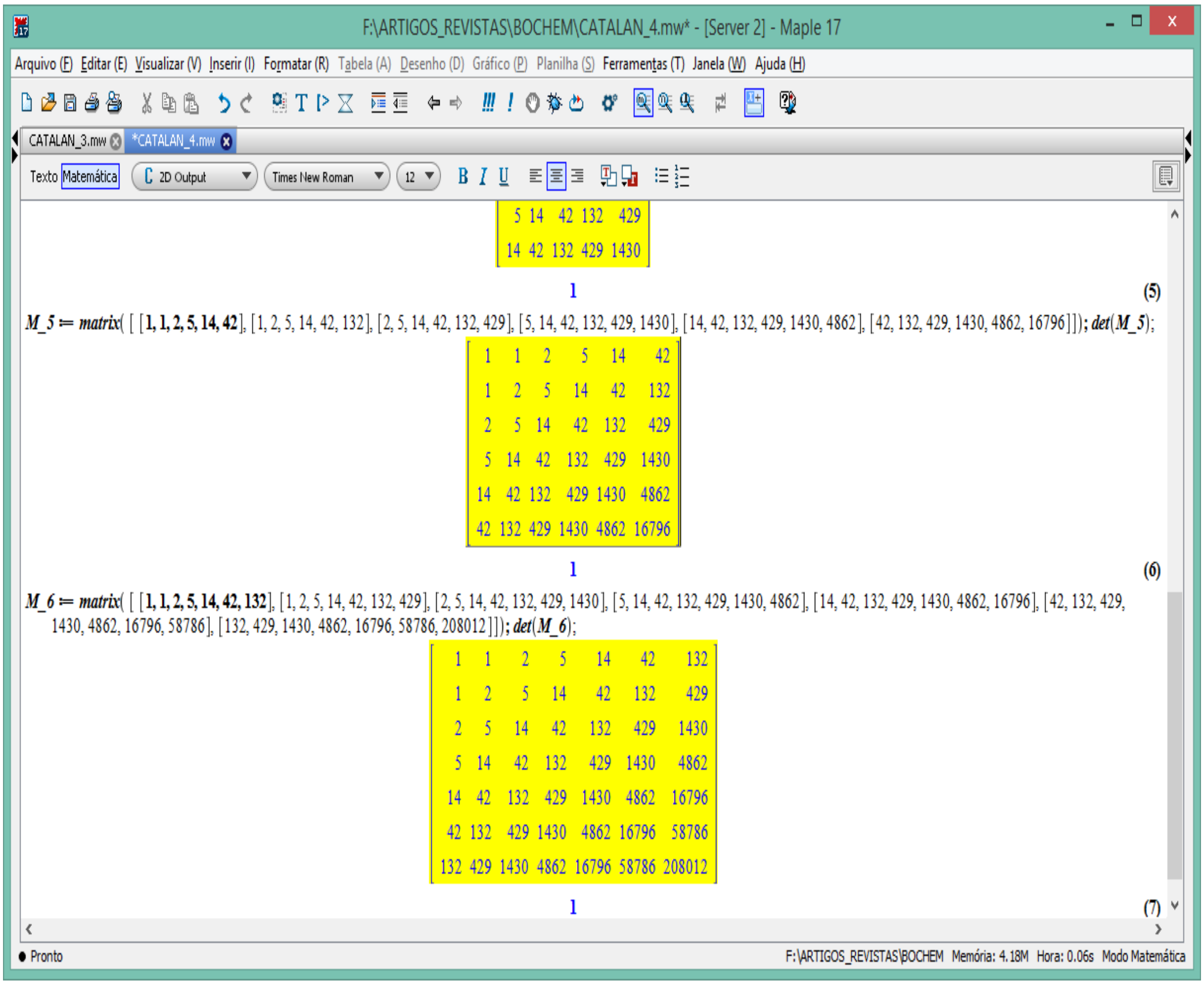

Figure 7. *With CAS Maple we can explore numerical and combinatorial relations derived from the matrix representations originated from Generalized Catalan numbers.*

5. Conclusion

In this work we will discuss some elementary properties on Catalan numbers. By means of a quick description of the professional mathematicians of the past we point out that E. Catalan was the mathematician who made known a subject of study that still preserves the interest of the current research on the subject and that contributed to the generalization of the numbers of Catalan, including the Fuss-Catalan numbers (Andrews, 1971) and Catalan's bivariate numbers (Bernhart, 1999; Buescu, 2010; Koshy & Salmassi, 2006; Gessell 1992; Gessel & Xin, 2004). On the other hand, we have questioned in Brazil several HM books that usually point out the historical aspects of concepts distanced from school reality and basic Mathematics. Thus, with the adoption of the presuppositions of a Didactic Engineering, above all, a didactic engineering addressed to the teacher of Mathematics in a context of historical research, we emphasize some properties whose matrix and combinatory representations allow their exploitation in the context of school teaching (see lemma 1, 2 and lemma 3(*)). By this way, we suggest an approach and a teaching perspective affected by the understanding of the progress and systematization of the mathematical models and the deep knowledge of numbers on the part of the teacher, including his multiple conceptual relations and generalizations.

Thus, in the previous sections, we have shown that several properties derived from the numbers of catala and their generalization can be explored in the context of teaching using technology. In the figures of the previous section we have approached some examples that, by means of simple commands of the software allow the appreciation of a great amount of elements and invariant properties pertaining to the numbers of Catalan.

We emphasize the approach of some formal mathematical definitions that confirm a research and the current interest in the process of generalization of Catalan numbers. Thus, in the predecessor sections, we provide the reader or possibly the mathematics teacher with certain properties that are the object of current interest and confirm the unstoppable, evolving character and ubiquity of Catalan numbers. This perception is important for the teacher's understanding in the context of school education.

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